

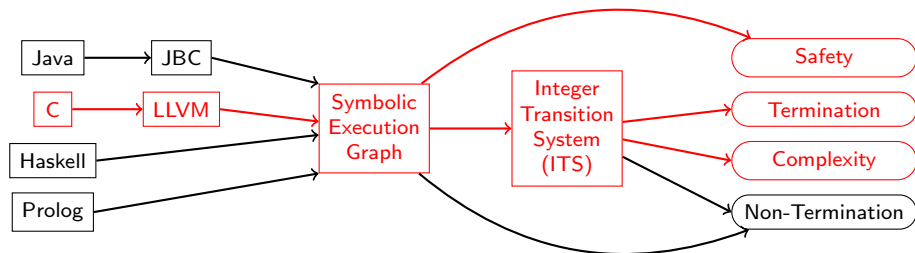
# Complexity Analysis for Bitvector Programs

Jürgen Giesl

LuFG Informatik 2, RWTH Aachen University, Germany

joint work with Jera Hensel and Florian Frohn

# Termination and Complexity Analysis in AProVE



- Handling of C programs with explicit pointer arithmetic
- **Drawback:**  
many tools assume mathematical integers  $\mathbb{Z}$  instead of bitvectors

# Mathematical Integers $\mathbb{Z}$ vs. Bitvectors

```
void f(unsigned int x) {  
    unsigned int j = 0;  
    while (j <= x) j++;  
}
```

for  $\mathbb{Z}$ :                    termination  
for bitvectors:            non-termination

```
void g(unsigned int j) {  
    while (j > 0) j++;  
}
```

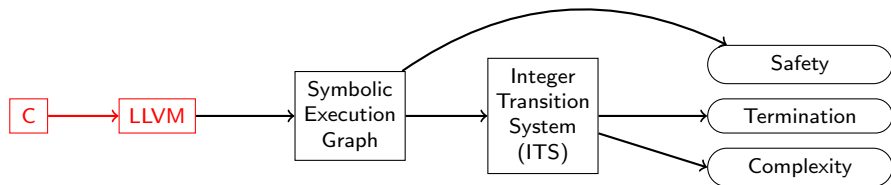
for  $\mathbb{Z}$ :                    non-termination  
for bitvectors:            termination

- **Goal:** adapt byte-accurate symbolic execution to bitvector arithmetic
- **Solution:** express bitvector relations by relations on  $\mathbb{Z}$ 
  - standard **SMT solving over  $\mathbb{Z}$**     for **symbolic execution**
  - standard **ITSs over  $\mathbb{Z}$**             for **termination and complexity analysis**

# From C to LLVM

```
define void @g(i32 j) {  
entry: 0: ad = alloca i32  
      1: store i32 j, i32* ad  
      2: br label cmp  
cmp:   0: j1 = load i32* ad  
      1: j1p = icmp ugt i32 j1, 0  
      2: br i1 j1p, label body,  
          label done  
body:  0: j2 = load i32* ad  
      1: inc = add i32 j2, 1  
      2: store i32 inc, i32* ad  
      3: br label cmp  
done:  0: ret void }  
}
```

```
void g(unsigned int j) {  
    while (j > 0) j++;  
}
```



# Abstract States

```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
          label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```

$a$	$(\text{entry}, 2)$	$\Leftarrow pos$
	$\{j = v_j, ad = v_{ad}\}$	$\Leftarrow PV$
	$\{[v_{ad}, v_{end}]\}$	$\Leftarrow AL$
	$\{v_{end} = v_{ad} + 3\}$	$\Leftarrow KB$
	$\{v_{ad} \xrightarrow{i32} v_j\}$	$\Leftarrow PT$

**Abstract state  $a$ :** *ERR* or

*pos*: program position (block, next instruction)

*PV*: program variables  $\rightarrow$  symbolic variables

*AL*: allocation list  $[v_1, v_2]$

*KB*: knowledge base (FO-(in)equalities over symbolic variables)

*PT*: points-to atoms  $v_1 \xrightarrow{\text{type}} v_2$

# Abstract States

```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
          label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```

$a$	$(\text{entry}, 2)$	$\Leftarrow pos$
	$\{j = v_j, ad = v_{ad}\}$	$\Leftarrow PV$
	$\{\llbracket v_{ad}, v_{end} \rrbracket\}$	$\Leftarrow AL$
	$\{v_{end} = v_{ad} + 3\}$	$\Leftarrow KB$
	$\{v_{ad} \xrightarrow{i32} v_j\}$	$\Leftarrow PT$

- $\langle a \rangle$ : FO formula containing
  - $KB$  and consequences of  $AL$  and  $PT$
  - information on ranges of integers:

$$j \text{ has type } i32 \quad \Rightarrow \quad 0 \leq \underbrace{PV(j)}_{v_j} \leq \underbrace{umax_{32}}_{2^{32}-1}$$

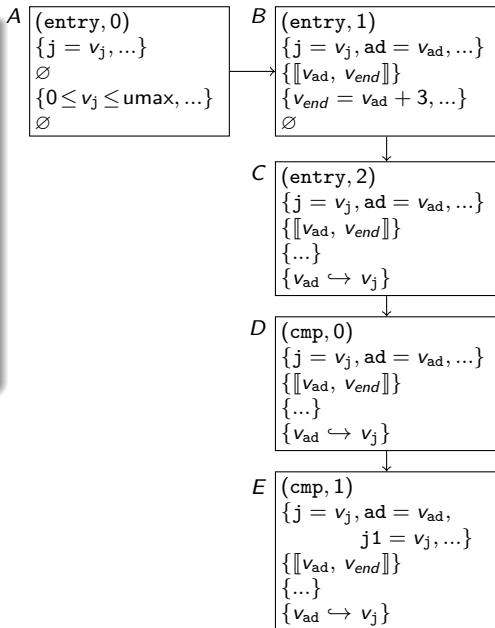
# Symbolic Execution

```
define void @g(i32 j) {  
entry: 0:ad = alloca i32  
      1:store i32 j, i32* ad  
      2:br label cmp  
cmp:   0:j1 = load i32* ad  
      1:j1p = icmp ugt i32 j1, 0  
      2:br i1 j1p, label body,  
        label done  
body:  0:j2 = load i32* ad  
      1:inc = add i32 j2, 1  
      2:store i32 inc, i32* ad  
      3:br label cmp  
done:  0:ret void }
```

A	(entry, 0)	← pos
	{j = v <sub>j</sub> , ...}	← PV
	∅	← AL
	{0 ≤ v <sub>j</sub> ≤ umax, ...}	← KB
	∅	← PT

# Symbolic Execution

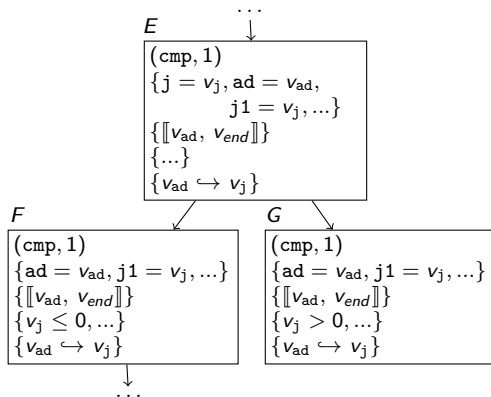
```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
           label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```





# Integer Comparison

```
define void @g(i32 j) {
entry:0:ad = alloca i32
      1:store i32 j, i32* ad
      2:br label cmp
cmp:   0:j1 = load i32* ad
      1:j1p = icmp ugt i32 j1, 0
      2:br i1 j1p, label body,
        label done
body:  0:j2 = load i32* ad
      1:inc = add i32 j2, 1
      2:store i32 inc, i32* ad
      3:br label cmp
done:  0:ret void }
```

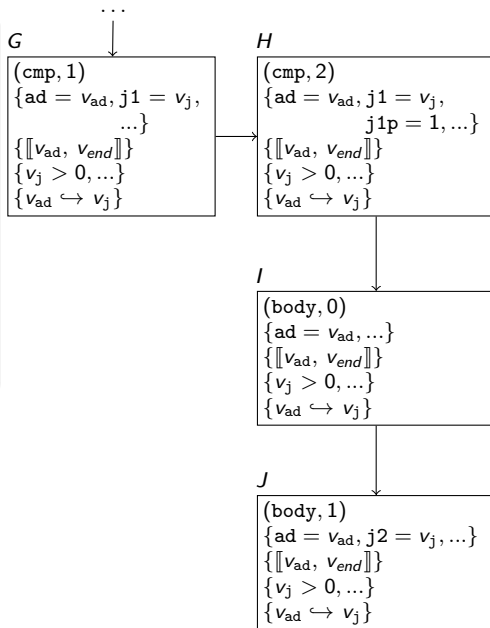


Symbolic execution rule for `x = icmp ugt i32 t1, t2`

- set `x` to 1 if  $\models \langle a \rangle \implies (PV(t_1) > PV(t_2))$
- set `x` to 0 if  $\models \langle a \rangle \implies (PV(t_1) \leq PV(t_2))$
- otherwise: case analysis

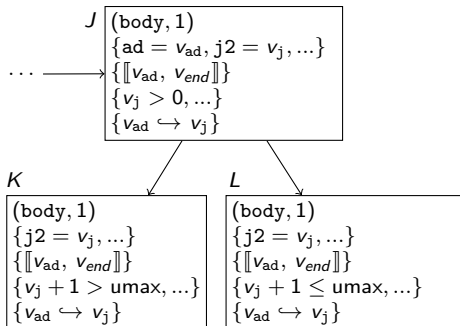
# Symbolic Execution

```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
           label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```



# Addition

```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
           label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```

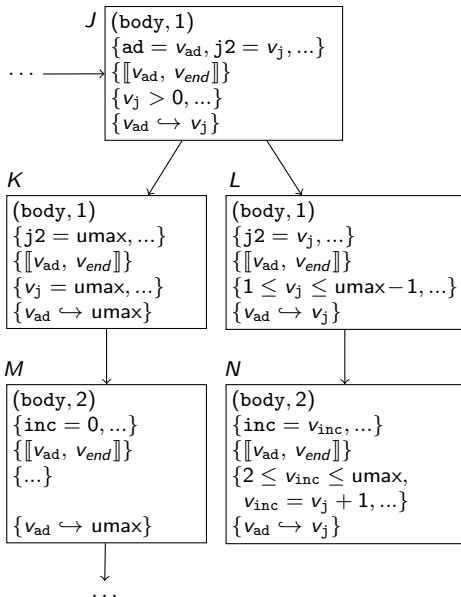


## Symbolic execution rule for $x = \text{add i32 } t_1, t_2$

- set  $x$  to  $PV(t_1) + PV(t_2)$  if  $\models \langle a \rangle \implies PV(t_1) + PV(t_2) \leq \text{umax}_{32}$
- set  $x$  to  $PV(t_1) + PV(t_2) - 2^{32}$  if  $\models \langle a \rangle \implies PV(t_1) + PV(t_2) > \text{umax}_{32}$
- otherwise: case analysis

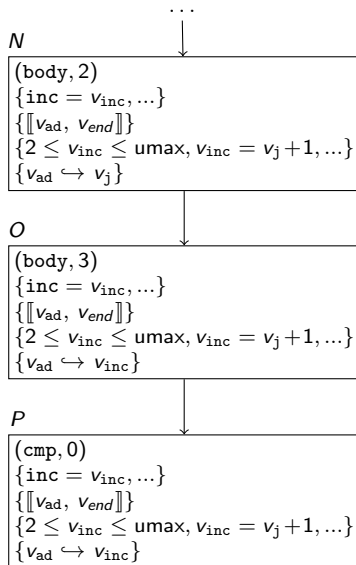
# Addition

```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
           label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```



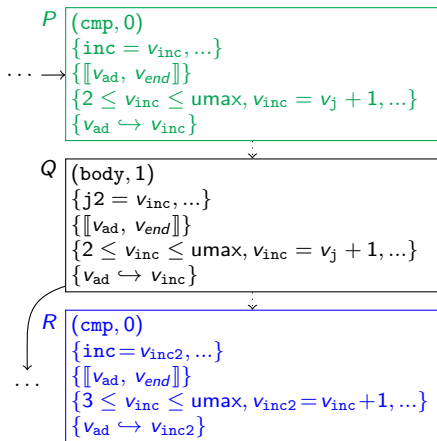
# Symbolic Execution

```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
           label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```



# Generalization

```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
           label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```

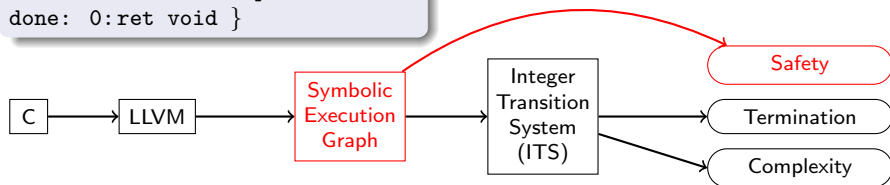
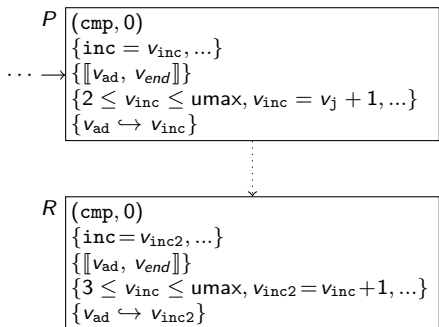


*P* is generalization of *R* with  $\mu(v_j) = v_{inc}$ ,  $\mu(v_{inc}) = v_{inc2}$

- $\mu(PV_P(x)) = PV_R(x)$  for all program variables  $x$
- $[[v_1, v_2]] \in AL_P$  implies  $[[\mu(v_1), \mu(v_2)]] \in AL_R$
- $\models \langle R \rangle \implies \mu(KB_P)$
- $v_1 \leftrightarrow v_2 \in PT_P$  implies  $\mu(v_1) \leftrightarrow \mu(v_2) \in PT_R$

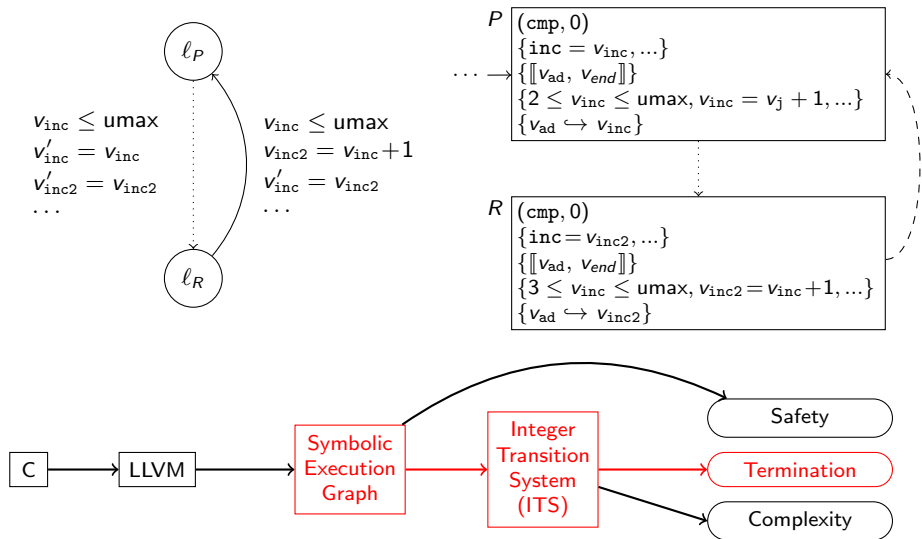
# Safety

```
define void @g(i32 j) {
entry: 0: ad = alloca i32
      1: store i32 j, i32* ad
      2: br label cmp
cmp:   0: j1 = load i32* ad
      1: j1p = icmp ugt i32 j1, 0
      2: br i1 j1p, label body,
          label done
body:  0: j2 = load i32* ad
      1: inc = add i32 j2, 1
      2: store i32 inc, i32* ad
      3: br label cmp
done:  0: ret void }
```



- Symbolic execution graph **complete** if leaves correspond to return
- Complete symbolic execution graph without *ERR*  $\implies$  **Safety**

# Termination



- ITS from cycles of symbolic execution graph
- ITS termination by existing tools  $\implies$  LLVM program terminates



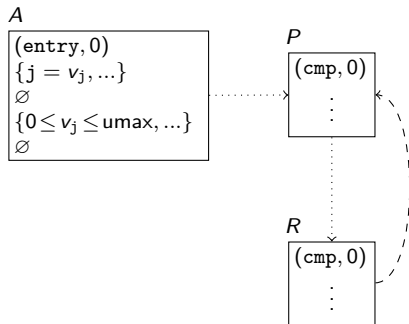
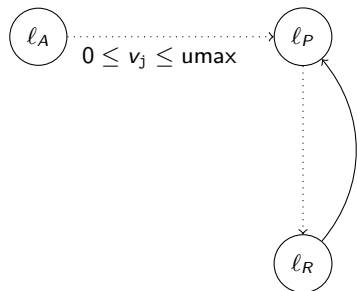
# Complexity

- Adapt transformation from symbolic execution graph to ITSs
- Use existing complexity tools for ITSs (over  $\mathbb{Z}$ )
- Arithmetic programs  $\in \mathcal{O}(1) \Rightarrow$  asymptotic complexity meaningless
  - $m$  instructions
  - $k$  variables  $x_1 : in_1, \dots, x_k : in_k$
  - runtime bounded by  $m \cdot 2^{n_1} \cdot \dots \cdot 2^{n_k}$
- **Goal:** infer concrete bounds  $\leq m \cdot 2^{n_1} \cdot \dots \cdot 2^{n_k}$ 

bounds depending on program's input parameters are better than  
bounds depending on sizes of types  $in$

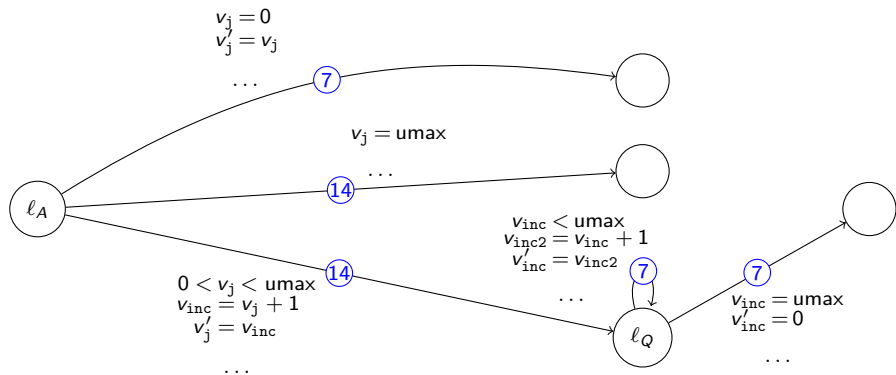
# Complexity: Adapt Approach for Termination

- 1 Every execution step counts for complexity  
⇒ generate ITS from whole graph, not just from cycles
- 2 Find bound on length of ITS evaluations  
depending on values in initial state



# Complexity: Adapt Approach for Termination

- 3 Simplify ITSs by filtering away variables and compressing transitions  
⇒ use **weighted** transitions

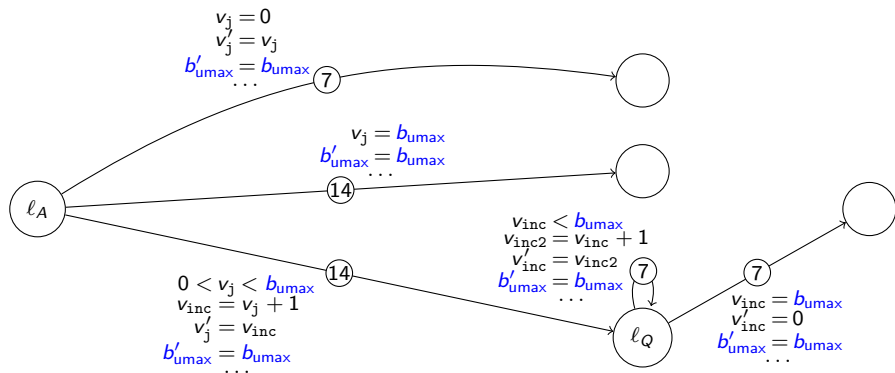


# Complexity: Adapt Approach for Termination

- 4 ITS tools prefer small asymptotic bounds

(huge constant **umax** preferred to bound depending on input parameters  $v_j$ )

⇒ replace size constant **umax** by variable  $b_{umax}$

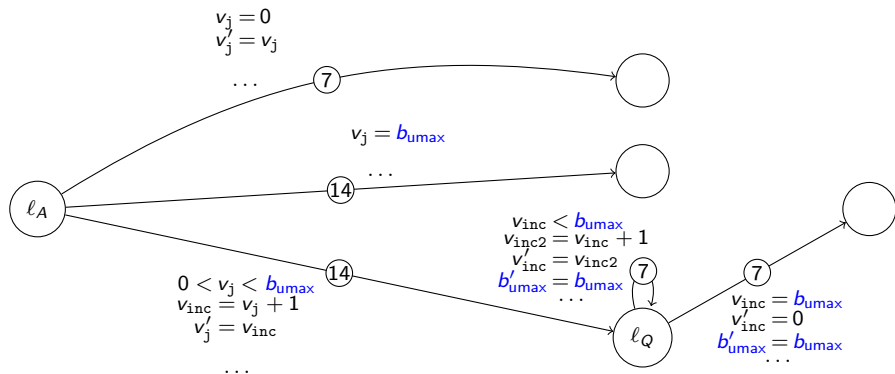


# Complexity: Adapt Approach for Termination

- 5 Bounds depending on program's input parameters  $v_j$  are better than bounds depending on sizes of types  $b_{umax}$

⇒ assign  $b_{umax}$  non-deterministically in initial transitions

⇒ if this fails, add  $b'_{umax} = b_{umax}$  in initial transitions

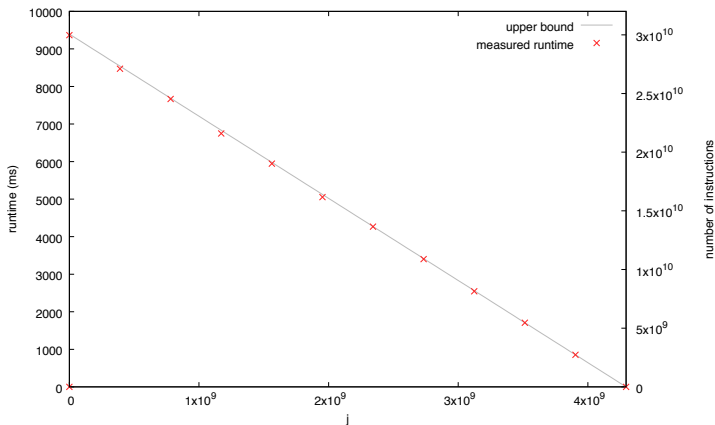


# Complexity: Adapt Approach for Termination

KoAT and CoFloCo yield:

$$\max(21, \underbrace{7 \cdot u_{\max}}_{2^{32}-1} - 7 \cdot j + 14)$$

```
void g(unsigned int j) {  
    while (j > 0) j++;  
}
```



# Complexity Analysis for Bitvector Programs

- Handling of **bitvectors** during **symbolic execution**
- Representation of **bitvectors** by **relations on  $\mathbb{Z}$** 
  - ⇒ standard SMT solving over  $\mathbb{Z}$
  - ⇒ standard back-end tools for termination and complexity analysis of ITSs
- Generation of **ITSs** slightly different for **termination** or **complexity**
- Implementation in **AProVE** (using **KoAT** and **CoFloCo** in the back-end)
  - **118** bitvector C programs from evaluations of other termination tools
  - **95** programs: **AProVE** proves termination
  - **60** programs: **AProVE** infers upper bound
    - **7** programs: small constant bound
    - **41** programs: linear or quadratic in input variables
    - **12** programs: bound also depends on size of types