Inference of Linear Upper-Bounds on the Expected Cost by Solving Cost Relations

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Overview

Cost Analysis

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Application of static techniques in order to determine the amount of resources that are needed to execute a program.

- Runtime (execution steps)
- Memory consumption
- Generic cost model: ticks

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- **Runtime**: Sort

```
for (i = 0; i < length(a); i++){
  val = a[i];
  j = i-1;
  while(j > 0 && a[j] > val){
    a[j+1] = a[j];
    j = j-1;
    tick(1);
  }
  a[j+1] = val;
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```

- **Worst case cost**:

$$\frac{\text{length}(a)^2 + \text{length}(a)}{2}$$

- **Memory**: Dynamic Array

```
while(! empty(list)){
  if list[i] > 0{
    if (j>=n){
      enlarge(a,l,n);
      tick(2*l*n);
    }
    a[j] = head(list);
    list = tail(list);
    j = j+1;
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  i = i+1;
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- **Worst case cost**:

$$2 * l * n * (length(list))$$

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Static analyzer for automatically inferring upper/lower bounds on the worst/best-case resource consumption of ABS programs.
`http://costa.fdi.ucm.es/saco/web/`

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- Functional + imperative.
- Based on concurrent objects.

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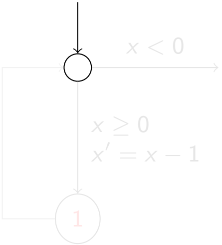

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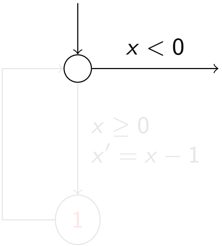

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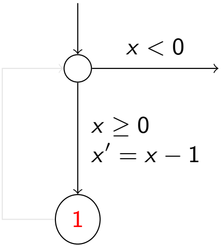

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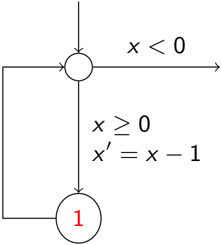

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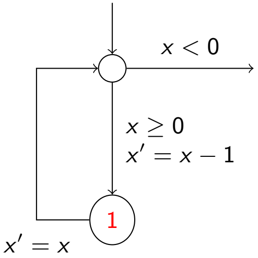

- **From CFG to Cost Relations**

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\begin{array}{ll}
C(x) = 0 & \{x < 0\} \\
C(x) = D(x') & \{x >= 0, x' = x - 1\} \\
D(x) = 1 + C(x') & \{x' = x\}
\end{array}
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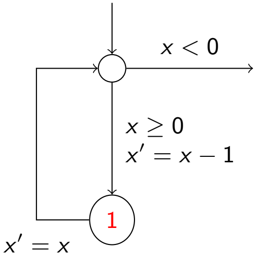

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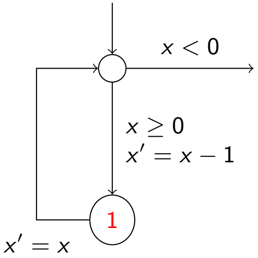

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Cost relations are like recurrence relations, but the have some difference like nondeterminism.

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while(x>=0){
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Upper-bound on the set of values to which a query might evaluate. It is an upper bound of the worst case of the execution.

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When worst-case is not enough...

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while(msgs != null){
  if(send(head(data))){
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- What happens if we trust the quality of the network to some degree?

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Probabilistic programs

Addition of probabilistic choices to the set of instructions:

$$\text{inst}_1 \oplus_p \text{inst}_2$$

where

- $p \in [0, 1]$
- inst_1 and inst_2 will execute with probability p and $1 - p$, respectively.
- It can be generalized such that inst_i is executed with probability p_i, and $\sum_i p_i = 1$

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while(x < y){
  x = x − 1 ⊕_{1/3} y = y − 2;
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Expected cost

Packages transmission on a net

We have got a net for transmitting packages on which the probability of succeeding is $\frac{2}{3}$, and we want to transmit n packages.

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while(msgs != null){
  if(send(head(data)))
    msgs = tail(msgs);
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$\Longrightarrow$

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\begin{array}{l}
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- S be an infinite sequence of independent choices for the probabilistic operation.
- $\texttt{cost}(n, S)$ be the cost of the execution (with input n) when taking the choices indicated in S.
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The sum is over all possible sequences S.

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We want to introduce probabilities into SACO.

Steps

- Introduction of probabilities into CFGs.
- Development of techniques for inferring expected cost of CFGs via CRs.
- Specify probabilities directly in ABS.

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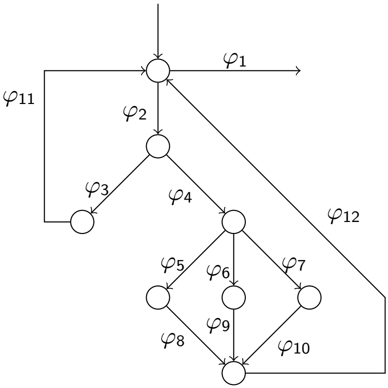
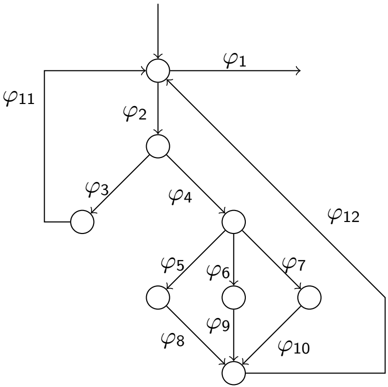


CFG with probabilities:

CFGs with probabilities

CFG without probabilities:



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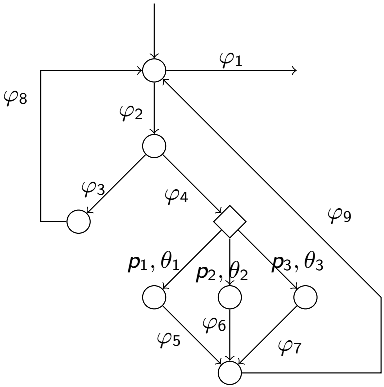

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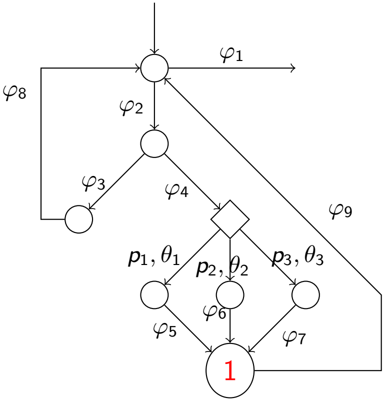

- Regular instructions represented by circles.
- Probability instructions represented by squares.
- Edges with updates and
  - Probabilities
  - Constraints
- Nodes with cost.
- $\sum_i p_i = 1$
- Choices in normal nodes cover the whole state space.

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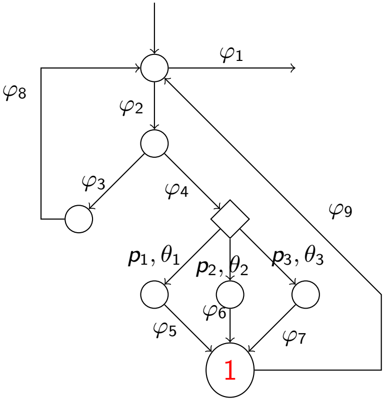

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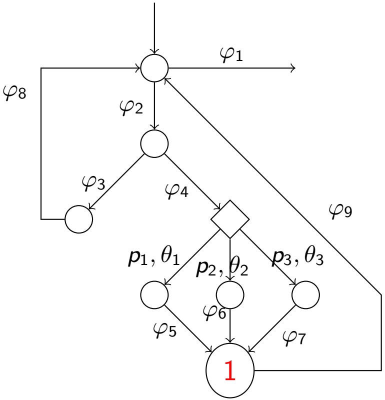

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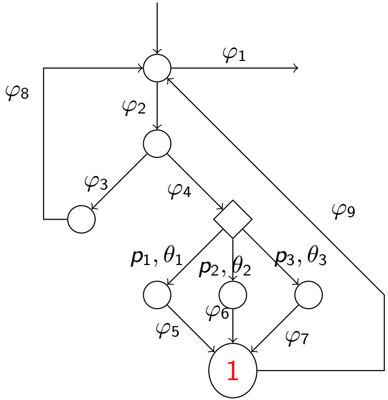

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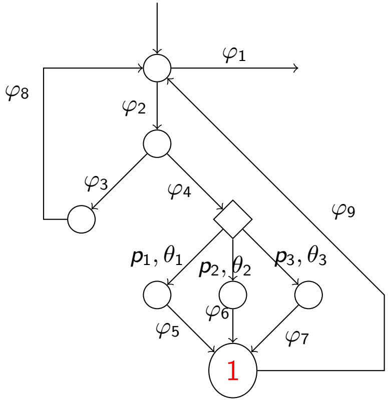

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while (n > 0){
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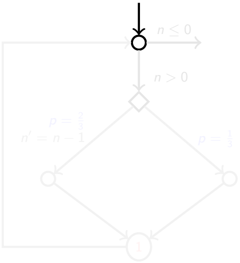

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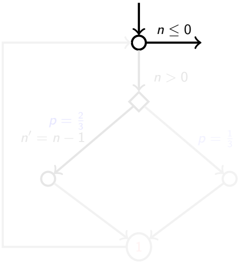

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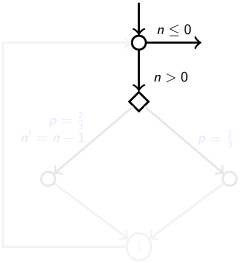

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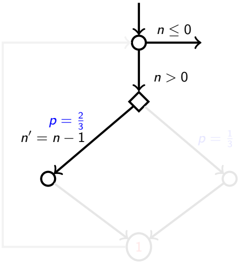

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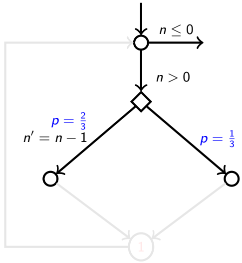

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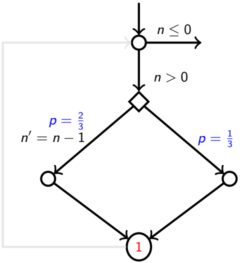

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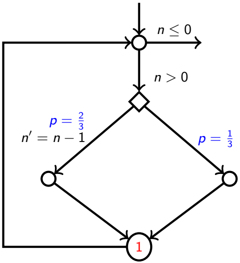

Generating CRs

- Each regular node $n \in N_r$ with outgoing transitions $n \xrightarrow{\varphi_i} n_i$, $1 \leq i \leq k$, contributes k cost relations of the form

$$C_n(\bar{x}) = \texttt{cost}(n) + C_{n_i}(\bar{x}') \qquad \varphi_i$$

- Each probabilistic node $n \in N_p$ with outgoing edges $n \xrightarrow{p_i,\theta_i} n_i$, for $1 \leq i \leq k$, contributes one cost relation

$$C_n(\bar{x}) = \texttt{cost}(n) + \sum_{i=1}^{k} p_i \cdot C_{n_i}(\bar{x}_i') \qquad \biguplus_{i=1}^{k} \theta_i$$

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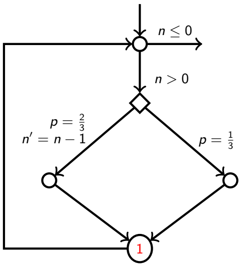

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\begin{array}{ll}
C(n) = 0 & \{n \le 0\} \\
C(n) = 1 + \dfrac{2}{3}D(n') + \dfrac{1}{3}D(n'') & \left\{n > 0,\ n' = n - 1,\ n'' = n\right\} \\
D(n) = C(n') & \left\{n' = n\right\}
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$$\Downarrow$$

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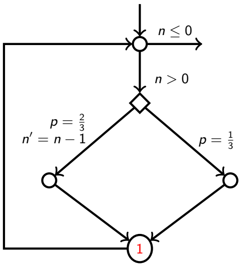

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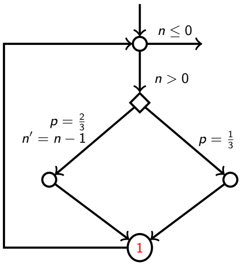

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Solving (probabilistic) CRs

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$$\forall n, \varphi_1 \implies f(n) \geq 0$$

$$\forall n, n', n'', \ \varphi_2 \implies f(n) \geq 1 + \frac{2}{3}f(n') + \frac{1}{3}f(n'')$$

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$$f(n) = a_1 n + a_0$$
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$$\forall n,\ \varphi_1 \implies a_1 n + a_0 \geq 0$$
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- We use Farkas Lemma to get constraints in terms of a_i from these equations.
- Solving these equations will give us the values for a_i.

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Solving (probabilistic) CRs

Upper bound functions must be non-negative.

- We introduce *max* function.
- If we denote

$$F(n) = \max(0, a_1 n + a_0)$$

then we will have

$$\begin{aligned} \forall n,\ \varphi_1 &\implies F(n) \geq 0 \\ \forall n, n', n'',\ \varphi_2 &\implies F(n) \geq 1 + \frac{2}{3}F(n') + \frac{1}{3}F(n) \end{aligned}$$

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$$\forall n,\ \varphi_1,\ a_1 n + a_0 < 0 \implies 0 \geq 0$$
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- Contraints on the left side of the implication lead us to non-linearity.
- We are in trouble with applying Farkas Lemma.

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Example: infinite expected cost

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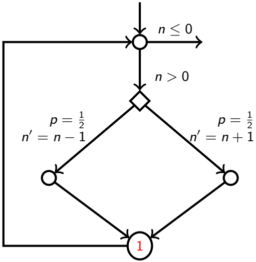

Multiphase example

```
while (x + 3 ≤ n){
   if (y < m)
      skip; ⊕_{1/2} y = y + 1;
   else
      skip; ⊕_{1/4} x = x + 1; ⊕_{1/4} x = x + 2; ⊕_{1/4} x = x + 3;
   tick(1);
}
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$$f(x, y, n, m) = \max(0, a_1 x + a_2 y + a_3 n + a_4 m + a_0)$$

If we use this global template... we will fail.

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We can start from a template that includes two parts (one for each phase).

$f(x, y, n, m) = \max(0, a_1 x + a_2 y + a_3 n + a_4 m + a_0) + \max(0, b_1 x + b_2 y + b_3 n + b_4 m + b_0)$

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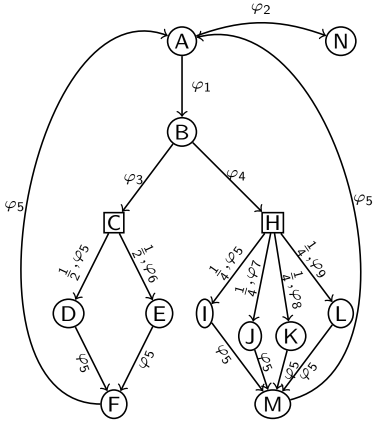

$$
\begin{aligned}
\varphi_1 &= \{x + 3 \leq n\} \cup \varphi_5 \\
\varphi_2 &= \{x + 3 > n\} \cup \varphi_5 \\
\varphi_3 &= \{y < m\} \cup \varphi_5 \\
\varphi_4 &= \{y' \geq m\} \cup \varphi_5 \\
\varphi_5 &= \{x' = x, n' = n, y' = y, m' = m\} \\
\varphi_6 &= \{x' = x, n' = n, y' = y + 1, m' = m\} \\
\varphi_7 &= \{x' = x + 1, n' = n, y' = y, m' = m\} \\
\varphi_8 &= \{x' = x + 2, n' = n, y' = y, m' = m\} \\
\varphi_9 &= \{x' = x + 3, n' = n, y' = y, m' = m\}
\end{aligned}
$$

Conclusions

- Our interest is to extend SACO to handle probabilistic programs
- We have extended the CFGs to have probabilities
- We have modeled the expected cost of CFGs with (probability) CRs.
- We have developed a technique for solving such CRs.
- Correctness: based on Farkas Lemma and Markov Decision Processes.