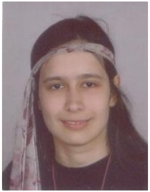
Improving Static Dependency Pairs for Higher-Order Rewriting

Carsten Fuhs



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Oxford, UK

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Why is termination of *higher-order* rewriting interesting?

Recently:

- Broader interest in termination analysis tools for higher-order rewriting
- Categories at the **Termination Competition**
  - "Higher-order rewriting union β": since 2010
  - "Rewriting with higher-order pattern matching on β-normal η-long forms (HRS)": under discussion
- New tools at TermComp 2017 (`SOL`) and 2018 (`SizeChangeTool`)
- Growing number of benchmarks in the **TPDB**
- Tools also used as back-end oracles in **Confluence Competition**

Higher-order rewriting in practice

- Specification language (with syntactic sugar) for **compiler synthesis**: CRSX [Rose, *RTA'11*] (successor tool `TransScript` used at IBM)

  `http://crsx.sourceforge.net/`

  Termination of higher-order specification
  $\Rightarrow$ **termination of the executable compiler**

- Termination analysis for functional programs

- Nuances in definitions of higher-order rewriting matter
  $\rightarrow$ discussion on `termtools` mailing list about
  plain matching vs matching *modulo beta*
  at TermComp 2018

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- allows transformations from many formalisms of interest for termination
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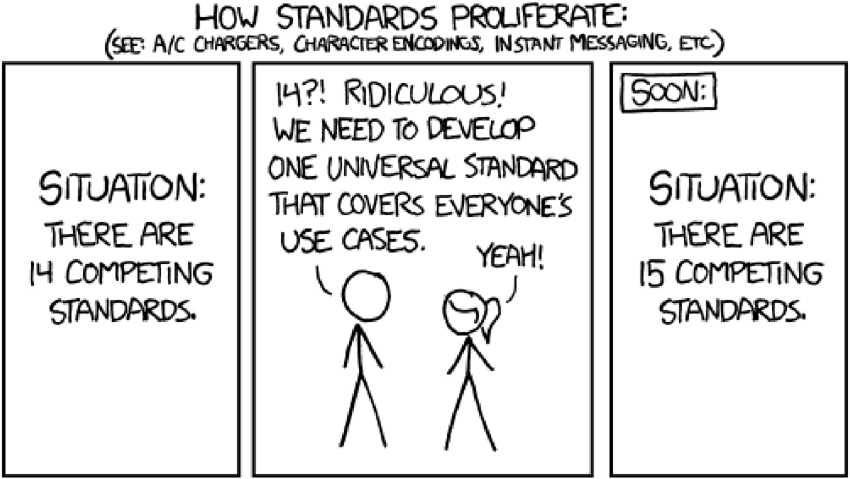

https://xkcd.com/927/

AFSMs: definition by example

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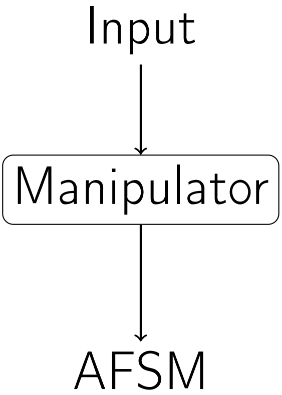

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Dependency Pairs and the Subterm Criterion

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- Extend to terms: $\nu(f^\sharp\ t_1\ \ldots\ t_n) = t_{\nu(f^\sharp)}$

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Subterm criterion [Hirokawa, Middeldorp, *IC'07*]:

- Use simple projection ν mapping each $\sharp$-symbol in $\mathcal{DP}$ to an argument

  $\nu(\mathsf{append}^\sharp) = 1$

- Extend to terms: $\nu(f^\sharp\ t_1\ \ldots\ t_n) = t_{\nu(f^\sharp)}$

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- If $s \rhd t$ for all $s \Rrightarrow t \in \mathcal{DP}$: termination proved!

Dependency Pairs and the Subterm Criterion

Example (Dependency Pair for append)

$\mathcal{DP}$: cons x xs $\rhd$ xs

Show termination using Dependency Pairs [Arts, Giesl, *TCS'00*]:

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Termination of Higher-Order Term Rewriting: map

Example (map)

$$\begin{array}{rcl} \mathsf{map}\ F\ \mathsf{nil} & \to & \mathsf{nil} \\ \mathsf{map}\ F\ (\mathsf{cons}\ x\ xs) & \to & \mathsf{cons}\ (F\ x)\ (\mathsf{map}\ F\ xs) \end{array}$$

$$0 : \mathsf{nat}, \quad \mathsf{s} : \mathsf{nat} \Rightarrow \mathsf{nat}, \quad \mathsf{nil} : \mathsf{list}, \quad \mathsf{cons} : \mathsf{nat} \Rightarrow \mathsf{list} \Rightarrow \mathsf{list},$$
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Static Dependency Pairs and the Subterm Criterion

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Termination of Higher-Order Term Rewriting: app-lam

Example ( )

$\mathcal{R}$:

$$\frac{\mathsf{app}\ (\mathsf{lam}\ F)\ x \ \rightarrow\ F\ x}{\mathsf{app} : \mathsf{o} \Rightarrow \mathsf{o} \Rightarrow \mathsf{o}, \qquad \mathsf{lam} : (\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o}}$$

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Alas, for $\omega = \mathsf{lam}\ (\lambda x.\mathsf{app}\ x\ x)$ we get

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Example (Encoding the Untyped Lambda-Calculus)

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So can't we just prohibit Static DPs for such malicious inputs?

Plain Function Passing (PFP)

Original soundness criterion for Static DPs [Kusakari, Isogai, Sakai, Blanqui, *TIS'09*] (adapted to our AFSMs):

Definition (Plain Function Passing (PFP))

A rewrite system $\mathcal{R}$ is Plain Function Passing (PFP)
if for all rules $f\ \ell_1\ \ldots\ \ell_n \to r \in \mathcal{R}$ and for all meta-variables F
of functional type in r, we have $F = \ell_i$ for some i.

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Let $\mathcal{R}$ be a rewrite system that is PFP. Then Static DPs are sound for proving termination of $\mathcal{R}$.

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- append $(\text{cons}\ x\ xs)\ ys \to \text{cons}\ x\ (\text{append}\ xs\ ys)$ ✓
- map $F\ (\text{cons}\ x\ xs) \to \text{cons}\ (F\ x)\ (\text{map}\ F\ xs)$

Plain Function Passing (PFP)

Original soundness criterion for Static DPs [Kusakari, Isogai, Sakai, Blanqui, *TIS'09*] (adapted to our AFSMs):

Definition (Plain Function Passing (PFP))

A rewrite system $\mathcal{R}$ is Plain Function Passing (PFP)
if for all rules $f\ \ell_1\ \ldots\ \ell_n \to r \in \mathcal{R}$ and for all meta-variables F of functional type in r, we have $F = \ell_i$ for some i.

Theorem (Static DPs sound for Plain Function Passing $\mathcal{R}$)

Let $\mathcal{R}$ be a rewrite system that is PFP. Then Static DPs are sound for proving termination of $\mathcal{R}$.

Which of the following rules is PFP?

- append (cons x xs) $ys \to$ cons x (append xs ys) ✓
- map F (cons x xs) $\to$ cons (F x) (map F xs) ✓
- app (lam F) $x \to F\ x$

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PFP improved in [Suzuki, Kusakari, Blanqui, *IPSJ TOP'11*].

Termination of Higher-Order Term Rewriting: ord-rec

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Idea: decrease in a **well-founded order** on types when we apply an F taken out of a left-hand side

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$\to$ Accessibility, see General Schema [Blanqui, Jouannaud, Okada, *TCS'02*], Computability Path Ordering [Blanqui, Jouannaud, Rubio, *LMCS'15*], and Computability Closure [Blanqui, *TCS'16*]

PFP with Accessibility: Accessible Function Passing

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$\Rightarrow 1 \in Acc(\mathsf{lim})$

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Related criterion (incomparable): [Blanqui, *WST'06*]

Static DPs and the Computable Subterm Criterion

Example (Static DPs for ord-rec)

$$\begin{array}{rcl} \mathsf{rec}^\sharp\ (\mathsf{s}\ X)\ K\ F\ G & \Rrightarrow & \mathsf{rec}^\sharp\ X\ K\ F\ G \\ \mathsf{rec}^\sharp\ (\mathsf{lim}\ H)\ K\ F\ G & \Rrightarrow & \mathsf{rec}^\sharp\ (H\ M)\ K\ F\ G \end{array}$$

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Candidate for Subterm Criterion: $\nu(\mathsf{rec}^\sharp) = 1$.

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- $\mathsf{s}\ X \quad \rhd \quad X$

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Static DPs and the Computable Subterm Criterion

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$\Rightarrow$ ord-rec is terminating by **Computable Subterm Criterion**

Conclusion

Here:

- Make Static DPs applicable to larger class of higher-order systems
- New class of infinite dependency chains to analyse: $\mathcal{R}$-computable chains
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Further extensions available:

https://arxiv.org/abs/1805.09390

- Unified framework integrating static and dynamic DPs
(dynamic DPs include DPs for calls to meta-variables of functional type)
- Many additional termination proving techniques as DP processors
- Completeness results wrt non-termination for static and dynamic DPs

Bonus slide: Plain Function Passing, v2

PFP: Must the F really be direct arguments on left-hand side?

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Example (deriv)

$$\frac{\mathsf{deriv}\ (\lambda x.\mathsf{sin}\ F[x])\ Y \to \mathsf{times}\ (\mathsf{deriv}\ (\lambda x.F[x])\ Y)\ (\mathsf{cos}\ F[Y])}{\mathsf{sin}, \mathsf{cos} : \mathsf{real} \Rightarrow \mathsf{real}, \qquad \mathsf{times} : \mathsf{real} \Rightarrow \mathsf{real} \Rightarrow \mathsf{real}, \\ \mathsf{deriv} : (\mathsf{real} \Rightarrow \mathsf{real}) \Rightarrow \mathsf{real} \Rightarrow \mathsf{real}}$$

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